TUGlab: A cooperative game theory toolbox

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TUGlab is a Matlab toolbox that covers the general topics on the theory of cooperative games with transferable utility. Game theory is a branch of mathematics which is used in modelling situations in which players with conflicting interests interact. Coalitional games are games in which the possibilities of the players are described by the available resources of different groups (coalitions) of players.

A cooperative game with transferable utility, or TU-game, is described by a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of agents and $v\colon 2^N \to \mathbb{R}$ is the characteristic function assigning to each coalition $S \subset N$ a value $v(S)$ representing the benefits from cooperation ($v(\emptyset) = 0$). A TU-game (N, v) is convex if for all $i \in N$ and all $S \subset T \subset N \setminus \{i\}$ it holds that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. Convexity implies that an agent contributes more to the benefits of a coalition when this coalition becomes larger. The imputation set is defined by

$$I(v) = \{x \in \mathbb{R}^n : x_i \geq v(\{i\})\ \forall i \in N, \sum_{i=1}^{n} x_i = v(N)\}.$$

It is the set of all individually rational and efficient allocations. The core of a TU-game (N, v) is defined by

$$C(v) = \{x \in I(v) : x(S) = \sum_{i \in S} x_i \geq v(S), \forall S \subset N\}.$$

The core allocations provide the agents with an incentive to maintain the grand coalition. The games with nonempty core are called balanced.

Let $\sigma\colon N \to \{1, \ldots, n\}$ be an ordering of the agents and Π_N be the set of all orderings of N. The marginal vector with respect to the order $\sigma \in \Pi_N$ is

$$m_i^\sigma = v(\{j \in N : \sigma(j) \leq \sigma(i)\}) - v(\{j \in N : \sigma(j) < \sigma(i)\}),\ i \in N.$$

The Weber set is the convex hull of all the marginal vectors

$$W(v) = \left\{ \sum_{\sigma \in \Pi_N} \alpha_\sigma m^\sigma : \alpha_\sigma \geq 0, \forall \sigma \in \Pi_N; \sum_{\sigma \in \Pi_N} \alpha_\sigma = 1 \right\}.$$

The core is always a subset of the Weber set, $C(v) \subset W(v)$. Moreover, $C(v) = W(v)$ if and only if the game (N, v) is convex.

An allocation rule assigns to each TU-game (N, v) exactly one allocation. The Shapley value $\phi(v)$ assigns to each agent i his expected marginal contribution, presuming that each of the $n!$ orders occurs equally likely: $\phi(v) = \frac{1}{n!} \sum_{\sigma \in \Pi_N} m^{\sigma}$. Let (N, v) be a TU-game such that $I(v) \neq \emptyset$. Given an allocation x the complaint, or excess, of coalition S is defined by $e(x, S) = v(S) - x(S)$. Next, let $e(x)$ denote the vector of excesses with its elements arranged in decreasing order. The nucleolus is the imputation $\eta \in I(v)$ such that $e(\eta) \leq_{\text{lex}} e(x)$, for all $x \in I(v)$. Let (N, v) be a balanced TU-game. The core-center is defined as the expectation $\mu(v) = E(U)$, being U the uniform distribution over the core $C(v)$.

The theory of games with transferable utility has applications to Operations Research, Economics as well as to other fields of social science such as political science. This is the subject of many undergraduate and graduate courses for students of Mathematics, Economics, Social Sciences, etc. TUGlab was born to be a complement to the books and other materials used in introductory courses on cooperative game theory, for example [1], [2], [4] and [5]. It has being designed to emphasize the geometrical aspects of the theory, to give both the instructor and the student a tool to compute and visualize basic concepts for any given 3 or 4 persons TU games, the only ones for which the imputation set, the Weber set and the core can be drawn. Though TUGlab is mainly a teaching tool, it has grown to include some specialized features that cover advanced topics of genuine interest for researchers.

Matlab is a high-level language and interactive environment that enables you to perform computationally intensive tasks, see [3]. The TUGlab toolbox is a collection of more than 50 Matlab functions to compute and visualize basic concepts for any given 3 or 4 persons TU game. The TUGlab package includes:

1. The main scripts (29 files) defining the functions concerning game theory concepts.
2. The auxiliary scripts (27 files) necessary for the computations but not directly related to game theory.
3. The data files (2 files with extension .mat).
4. The User's Guide in PDF format.
5. Some tutorials, with several examples that illustrate how the toolbox can be used including several animations and a library with many interesting examples of well-known games.

The TUGlab toolbox runs on any implementation of the later releases of the Matlab product: Matlab 6 and Matlab 7 on Unix, PC or Macintosh. To install it, just paste the package on your hard disk, choosing a directory that belongs to the Matlab search path. The whole package can be downloaded from the web page of the Santiago Game Theory (SaGaTh) Group: `http://eio.usc.es/pub/io/xogos` or directly from the site

`http://webs.uvigo.es/matematicas/campus_vigo/profesores/mmiras/TUGlabWeb/TUGlab.html`

Among the topics cover by TUGlab we can highlight: checking monotonicity, convexity or balancedness of a game; computing value solutions as the Shapley value, the nucleolus or the τ-value; drawing the imputation set, the Weber set and the core of a game; returning the Harsanyi dividends, the multi-linear extension and the normalization of any given game,...

The TUGlab functions run directly from the Matlab Command Window. The typical syntax of a command is **S=function(A)**, where A is the characteristic function of the game that must be introduce as a vector A=[v(1) v(2) v(3) v(12) v(13) v(23) v(123)], for 3 persons games, or A=[v(1) v(2) v(3) v(4) v(12) v(13) v(14) v(23) v(24) v(34) v(123) v(124) v(134) v(234) v(1234)], for 4 persons games; $function$ is the name of the TUGlab command, and S is the outcome. So, for example, the next commands

```
A=[0 0 0 100 200 300 400];[control,info]=convexgame(A)
```

produce the outcome

```
control = 0
info = v{123}-v{23}<v{13}-v{3}
```

which tells us that the game given by A is not convex because the inequality $v(123) - v(23) < v(13) - v(3)$ holds. To draw the core of a game A it is optimal to plot the imputation set first and then superimpose the core. The following commands

```
A=[0,0,0,0,10,40,30,60,10,20,90,90,120,130,160];
clf, imputationset(A), hold on, axis(axis)
coreset(A)
```

produce a picture of the core of game A: a convex polytope in a three dimensional space. Additionally, you can compute the vertices of the core polytope, check if they are marginal vector, see if the Shapley value belongs to the core, etc.

Please, report any bugs or suggestions about TUGlab to the authors. All comments would be welcome. We would like to thank the members of the Santiago Game Theory (SaGaTh) Group for their feedback and support. This work has being financially sponsored by Xunta de Galicia, grant PGIDT00PXI30001PN, and the Spanish Ministry of Education, grant SEJ2005-07637-C02-02.

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